Higher-order tangent and secant numbers

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Abstract. In this paper higher-order tangent numbers and higher-order secant numbers, $\{\mathscr{T}(n,k)\}_{n,k=0}^{\infty}$ and $\{\mathscr{S}(n,k)\}_{n,k=0}^{\infty}$, have been studied in detail. Several known results regarding $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ have been brought together along with many new results and insights and they all have been proved in a simple and unified manner. In particular, it is shown that the higher-order tangent numbers $\mathscr{T}(n,k)$ constitute a special class of the partial multivariate Bell polynomials and that $\mathscr{S}(n,k)$ can be computed from the knowledge of $\mathscr{T}(n,k)$. In addition, a simple explicit formula involving a double finite sum is deduced for the numbers $\mathscr{T}(n,k)$ and it is shown that $\mathscr{T}(n,k)$ are linear combinations of the classical tangent numbers T_n.

1. Introduction

Let us, for non-negative integers n and k, define the *n–th tangent number of order k*, $\mathscr{T}(n,k)$, by the generating relation (see [1, p. 259]; *cf.* [2, p. 428] and [3, p. 305])

$$\frac{\tan^k t}{k!} = \sum_{n=k}^{\infty} \mathscr{T}(n,k) \frac{t^n}{n!}. \qquad (1.1)$$

Similarly, define *n–th secant number of order k*, $\mathscr{S}(n,k)$, by (*cf.* [2, p. 428])

$$\frac{\sec t \tan^k t}{k!} = \sum_{n=k}^{\infty} \mathscr{S}(n,k) \frac{t^n}{n!}. \qquad (1.2)$$

Since $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ are, respectively, generalizations of the classical (and well-known) tangent (or, zag) numbers T_n (see [1, p. 259] and [4])

$$\tan t = \sum_{n=1}^{\infty} T_n \frac{t^n}{n!}, \qquad \text{i.e., } T_n := \mathscr{T}(n,1) \qquad (1.3)$$

and secant (or, zig) numbers S_n (see [4] and [5, p. 63])

$$\sec t = \sum_{n=0}^{\infty} S_n \frac{t^n}{n!}, \qquad \text{i.e., } S_n := \mathscr{S}(n,0), \qquad (1.4)$$

the numbers $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ may also be referred to as the higher-order (or, generalized) tangent numbers and the higher-order (or, generalized) secant numbers.

In a recent paper by Cvijović [6], very simple and compact closed-form higher derivative formulae are derived for eight trigonometric and hyperbolic functions [6, Corollaries 1 and 2] and they involve the numbers $T(n,k)$ and $S(n,k)$ given by

$$T(n,k) = \mathscr{T}(n,k)\,k! \quad \text{and} \quad S(n,k) = \mathscr{S}(n,k)\,k!. \tag{1.5}$$

The elegance and remarkable simplicity of the obtained results (see Propositions 11 and 12) have been the main motivation behind our further interest in these numbers. It has turned out that they are not sufficiently studied as yet and it is aimed here to thoroughly examine $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$. Several known results have been brought together along with many new results and insights and they all have been proved in a simple and unified manner (for further details, see Section 3).

2. Properties of numbers $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$

In what follows, it is assumed, unless otherwise indicated, that k, l, m and n are non-negative integers and we set an empty sum to be zero and

$$D_t^n := \frac{d^n}{dt^n} \qquad (n \geq 1).$$

We begin by an observation that it can be easily seen, from definitions (1.1) and (1.2) and after some parity considerations, that Proposition 1 holds true.

Proposition 1. *Let $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ be the numbers defined by (1.1) and (1.2). Then:* **a)** *$\mathscr{T}(n,k) \neq 0$ is only when $1 \leq k \leq n$ and either both n and k are even or both n and k are odd. In other words, $\mathscr{T}(2m, 2l+1) = 0$ and $\mathscr{T}(2m+1, 2l) = 0$;* **b)** *$\mathscr{S}(n,k) \neq 0$ is only when $0 \leq k \leq n$ and either both n and k are even or both n and k are odd. In other words, $\mathscr{S}(2m, 2l+1) = 0$ and $\mathscr{S}(2m+1, 2l) = 0$.*

Proposition 2. *Assume that $n, k \geq 1$ and let $\mathscr{T}(n,k)$ and T_n, respectively, be the higher-order tangent numbers and tangent numbers. Then, we have*

$$\mathscr{T}(k,k) = T_1^k \tag{2.1}$$

and, for $n \neq k$,

$$\mathscr{T}(n,k) = \frac{n+1}{n-k} \cdot \frac{1}{T_1} \sum_{r=1}^{n-k} \binom{n}{r} \left(\frac{k+1}{n+1} - \frac{1}{r+1} \right) T_{r+1}\, \mathscr{T}(n-r,k). \tag{2.2}$$

Demonstration. The proof is based on the following known formula for powers of series [7, Eqs. (1.1) and (3.2)]. For a fixed k, put

$$\left(\sum_{n=1}^{\infty} a_n t^n \right)^k = \sum_{n=k}^{\infty} b_n t^n \qquad (k \geq 1),$$

then the coefficients b_n, $n \geq k$, are given by $b_k = a_1^k$ and

$$b_n = \frac{1}{a_1(n-k)} \sum_{r=1}^{n-k} \Big[(r+1)(k+1) - (n+1) \Big] a_{r+1}\, b_{n-r} \quad (n \geq k+1).$$

It is not difficult to show that this result together with the definitions of the numbers $\mathscr{T}(n,k)$ and T_n in (1.1) and (1.3) yields (2.1) and (2.2).

Proposition 3. *Let $\mathscr{T}(n,k)$, $\mathscr{S}(n,k)$ and S_n, respectively, be the higher-order tangent numbers, the higher-order secant numbers and secant numbers. Then*

$$\mathscr{S}(n,k) = \sum_{n^*=0}^{n} \binom{n}{n^*} \mathscr{T}(n^*,k)\, S_{n-n^*}. \tag{2.3}$$

Demonstration. Recall that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two series, then their the Cauchy product is the series $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Hence, the required formula (2.3) is implied by the Cauchy product of power series expansions (1.4) and (1.1) for functions involving the product $\sec t \tan^k t/k!$ in (1.2).

Proposition 4. *Consider the multivariate (exponential) partial Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_n)$ defined by the formal power series expansion ([1, pp. 133–137] and [8, pp. 412–417])*

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_n)\frac{t^n}{n!} \tag{2.4}$$

and let $\mathscr{T}(n,k)$ and T_n be the higher-order tangent numbers and tangent numbers, respectively. Then

$$\mathscr{T}(n,k) = B_{n,k}(T_1, T_2, \ldots, T_n). \tag{2.5}$$

Demonstration. The proposed identity is readily available from (2.4) and (1.3) in conjunction with (1.1).

In view of Proposition 4, the higher-order tangent numbers constitute a special class of the Bell polynomials. Thus, for fixed n and k, $\mathscr{T}(n,k)$ is a homogenous and isobaric polynomial in $T_1, T_2, \ldots, T_n$ of total degree k and total weight n, i.e., it is a linear combination of monomials $T_1^{k_1} \cdot T_2^{k_2} \cdots T_n^{k_n}$ whose partial degrees and weights are constantly given by $k_1 + k_2 + \ldots + k_n = k$ and $k_1 + 2k_2 + \ldots + nk_n = n$.

Example 1. In order to demonstrate an application of Proposition 4, we tabulate several $\mathscr{T}(n,k)$ given in terms of T_n. Note that a list of $B_{n,k} \equiv B_{n,k}(x_1, x_2, \ldots, x_n)$ for $k \leq n \leq 12$ can be, for instance, found in [1, pp. 307–308], while the needed values of T_n are $T_1 = 1, T_3 = 2, T_5 = 16, T_7 = 272, T_9 = 7936$.

$$\mathscr{T}(1,1) = T_1,$$

$$\mathscr{T}(2,2) = T_1^2,$$

$$\mathscr{T}(3,1) = T_3, \quad \mathscr{T}(3,3) = T_1^3,$$

$$\mathscr{T}(4,2) = 4\,T_1 T_3, \quad \mathscr{T}(4,4) = T_1^4,$$

$$\mathscr{T}(5,1) = T_5, \quad \mathscr{T}(5,3) = 10\,T_1^2 T_3, \quad \mathscr{T}(5,5) = T_1^5,$$

$$\mathscr{T}(6,2) = 6\,T_1 T_5 + 10\,T_3^2, \quad \mathscr{T}(6,4) = 20\,T_1^3 T_3, \quad \mathscr{T}(6,6) = T_1^6,$$

$$\mathscr{T}(7,1) = T_7, \quad \mathscr{T}(7,3) = 21\,T_1^2 T_5 + 70\,T_1 T_3^2, \quad \mathscr{T}(7,5) = 35\,T_1^4 T_3, \quad \mathscr{T}(7,7) = T_1^7,$$

$$\mathscr{T}(8,2) = 8\,T_1 T_7 + 56\,T_3 T_5, \quad \mathscr{T}(8,4) = 56\,T_1^3 T_5 + 280\,T_1^2 T_3^2, \quad \mathscr{T}(8,6) = 56\,T_1^5 T_3,$$

$$\mathscr{T}(8,8) = T_1^8,$$

$$\mathscr{T}(9,1) = T_9, \ \mathscr{T}(9,3) = 36\,T_1^2 T_7 + 504\,T_1 T_3 T_5 + 280\,T_3^3, \ \mathscr{T}(9,5) = 126\,T_1^4 T_5 + 840\,T_1^3 T_3^3,$$

$$\mathscr{T}(9,7) = 84 T_1^6 T_3, \ \mathscr{T}(9,9) = T_1^9.$$

Corollary 1. *Let $\mathscr{T}(n,k)$ and T_n be the higher-order tangent numbers and tangent numbers, respectively. Then*

$$\text{a)}\quad \mathscr{T}(n+1,k+1) = \sum_{r=0}^{n-k}\binom{n}{r}T_{r+1}\,\mathscr{T}(n-r,k),\qquad\qquad (2.6)$$

$$\mathscr{T}(0,0) = 1,\qquad \mathscr{T}(n,0) = 0 \qquad (n \geq 1);$$

$$\text{b)}\quad \mathscr{T}(n+1,k+1) = \frac{1}{k+1}\sum_{r=0}^{n-k}\binom{n+1}{r+1}T_{r+1}\,\mathscr{T}(n-r,k),\qquad (2.7)$$

$$\mathscr{T}(0,0) = 1,\qquad \mathscr{T}(n,0) = 0 \qquad (n \geq 1).$$

Demonstration. The results are immediate consequences of Proposition 4 and the recurrence relations [8, p. 415, Eqs. (11.11) and (11.12)]

$$B_{n+1,k+1} = \sum_{r=0}^{n-k}\binom{n}{r}x_{r+1}\,B_{n-r,k}$$

and

$$B_{n+1,k+1} = \frac{1}{k+1}\sum_{r=0}^{n-k}\binom{n+1}{r+1}x_{r+1}\,B_{n-r,k}.$$

satisfied by the partial Bell polynomials $B_{n,k}$ with $B_{0,0} := 1$.

Proposition 5. *Let S_n be the secant numbers (1.4). The numbers $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ obey the following recurrence relations*

$$\text{a)}\quad \mathscr{T}(n+1,k) = \mathscr{T}(n,k-1) + k(k+1)\mathscr{T}(n,k+1) \quad (n \geq 0, k \geq 1),\qquad (2.8)$$

$$\mathscr{T}(0,0) = 1,\qquad \mathscr{T}(n,0) = 0 \qquad (n \geq 1);$$

$$\text{b)}\quad \mathscr{S}(n+1,k) = \mathscr{S}(n,k-1) + (k+1)^2\mathscr{S}(n,k+1) \quad (n \geq 0, k \geq 1),\qquad (2.9)$$

$$\mathscr{S}(n,0) = S_n \qquad (n \geq 0).$$

Demonstration. The recurrence relation (2.8) follows at once from

$$\frac{\tan^{k-1}t}{(k-1)!} + k(k+1)\frac{\tan^{k+1}t}{(k+1)!} = \frac{\sec^2 t\tan^{k-1}t}{(k-1)!} = \frac{1}{k!}D_t\tan^k t$$

and

$$\frac{1}{k!}D_t\tan^k t = \sum_{n+1=k}^{\infty}\mathscr{T}(n+1,k)\frac{t^n}{n!}.$$

Similarly, by making use of

$$\frac{1}{k!}D_t\big(\sec t\tan^k t\big) = \frac{\sec^3 t\tan^{k-1}t}{(k-1)!} + \frac{\sec t\tan^{k+1}t}{k!} = \sum_{n+1=k}^{\infty}\mathscr{T}(n+1,k)\frac{t^n}{n!}.$$

and

$$\frac{\sec t\tan^{k-1}t}{(k-1)!} + k(k+1)\frac{\sec t\tan^{k+1}t}{(k+1)!} = \frac{\sec^3 t\tan^{k-1}t}{(k-1)!},$$

it is straightforward to arrive at (2.9).

TABLE 1. Higher-order tangent numbers $\mathscr{T}(n,k)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	2	0	1						
4	0	0	8	0	1					
5	0	16	0	20	0	1				
6	0	0	136	0	40	0	1			
7	0	272	0	616	0	70	0	1		
8	0	0	3968	0	2016	0	112	0	1	
9	0	7936	0	28160	0	5376	0	168	0	1

For the sake of ready reference, by employing Proposition 5, we compute and list several of the numbers $\mathscr{T}(n,k)$ in Table 1 and several of the numbers $\mathscr{S}(n,k)$ in Table 2.

TABLE 2. Higher-order secant numbers $\mathscr{S}(n,k)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	1	0	1							
3	0	5	0	1						
4	5	0	14	0	1					
5	0	61	0	30	0	1				
6	61	0	331	0	55	0	1			
7	0	1385	0	1211	0	91	0	1		
8	1385	0	12284	0	3486	0	140	0	1	
9	0	50521	0	68060	0	8526	0	204	0	1

Proposition 6. *We have that:*

a) $$\sum_{k=1}^{n+1} (k-1)!\,\mathscr{T}(n+1,k) = \begin{cases} \left(2^n - 1\right)T_n & n \text{ odd} \\ 2^n S_n & n \text{ even} \end{cases} \quad ; \tag{2.10}$$

b) $$\sum_{k=0}^{n} k!\,\mathscr{T}(n,k) = 2^{n-1}\begin{cases} T_n & n \text{ odd} \\ S_n & n \text{ even} \end{cases} \quad . \tag{2.11}$$

Demonstration. Observe that nth derivative of $\tan z$ is a polynomial in $\tan z$, *i.e.* $D_t^n \tan z = P_n(\tan z)$, where $P_n(x)$ is explicitly given by   (*cf.* (1.3), (1.5) and [6, Eq. (3.5)])

$$P_n(x) = T_n + \sum_{k=1}^{n+1} (k-1)!\, \mathscr{T}(n+1, k)\, x^k.$$

Then, having in mind the identities $D_t^n \tan(\pi/4 + z/2) = 2^{-n} P_n\big(\tan(\pi/4 + z/2)\big)$ and $\tan(\pi/4 + z/2) = \tan z + \sec z$, by the last formula we obtain (2.10). Further, starting from the left-hand side of (2.10) and by applying the recurrence relation (2.8), we have

$$\sum_{k=1}^{n+1} (k-1)!\mathscr{T}(n+1, k) = \sum_{k=0}^{n} k!\mathscr{T}(n, k) + \sum_{k=2}^{n+2} k!\mathscr{T}(n, k) = 2\sum_{k=0}^{n} k!\mathscr{T}(n, k) - T_n,$$

which, by appealing to the right-hand side of (2.10), yields (2.11).

In Proposition 7 below we shall require the *higher-order arctangent numbers* $\mathscr{T}^*(n, k)$ defined by the generating relation [1, p. 260]

$$\frac{\arctan^k t}{k!} = \sum_{n=k}^{\infty} \mathscr{T}^*(n, k) \frac{t^n}{n!}, \tag{2.12}$$

or, equivalently, by the recurrence relation [1, p. 260]

$$\mathscr{T}^*(n+1, k) = \mathscr{T}^*(n, k-1) - n(n-1)\mathscr{T}^*(n-1, k) \quad (n \geq 0, k \geq 1), \tag{2.13}$$

$$\mathscr{T}^*(0, 0) = 1, \qquad \mathscr{T}^*(n, 0) = 0 \qquad (n \geq 1).$$

TABLE 3. Higher-order arctangent numbers $\mathscr{T}^*(n, k)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	−2	0	1						
4	0	0	−8	0	1					
5	0	24	0	−20	0	1				
6	0	0	184	0	−40	0	1			
7	0	−720	0	784	0	−70	0	1		
8	0	0	−8448	0	2464	0	−112	0	1	
9	0	40320	0	−52352	0	6384	0	−168	0	1

The numbers $\mathscr{T}^*(n, k)$ (see Table 3) are involved in the following formula which can be easily proved (*cf.* [12, Eqs. (3.4), (3.6) and (3.7)])

$$\tan^n t = \frac{1}{(n-1)!} \sum_{r=0}^{n-1} \mathscr{T}^*(n, r+1)\, D_t^r \tan t + \begin{cases} 0 & n \text{ odd} \\ (-1)^{\frac{n}{2}} & n \text{ even} \end{cases} \quad (n \geq 1). \tag{2.14}$$

Indeed, let us make use of the induction on n. In case $n = 1$ the formula (2.14) is clearly true. For the induction step, we assume that (2.14) is true for n and establish it for $n + 1$. This obviously follows, since by the recurrence relation (2.13) we have

$$\frac{1}{n!} \sum_{r=0}^{n} \mathscr{T}^*(n+1, r+1) D_t^r \tan t = \Delta_1 - \Delta_2 = \frac{1}{n} D_t \tan^n t - \tan^{n-1} t = \tan^{n+1} t,$$

where

$$n \Delta_1 = \frac{1}{(n-1)!} \sum_{r=1}^{n} \mathscr{T}^*(n, r) D_t^r \tan t = D_t \left(\frac{1}{(n-1)!} \sum_{r=0}^{n-1} \mathscr{T}^*(n, r+1) D_t^r \tan t \right)$$
$$= D_t \tan^n t$$

and

$$(n-2)! \Delta_2 = \sum_{r=0}^{n} \mathscr{T}^*(n-1, r+1) D_t^r \tan t = \sum_{r=0}^{n-2} \mathscr{T}^*(n-1, r+1) D_t^r \tan t$$
$$= (n-2)! \tan^{n-1} t.$$

Proposition 7. *Let T_n and $\mathscr{T}^*(n, k)$ be the tangent numbers and the higher-order arctangent numbers. Then, the higher-order tangent numbers $\mathscr{T}(n, k)$ are given by*

$$\mathscr{T}(n, k) = \frac{1}{k!} \cdot \frac{1}{(k-1)!} \sum_{r=0}^{k-1} T_{n+r} \, \mathscr{T}^*(k, r+1) \quad (n, k \geq 1). \tag{2.15}$$

Demonstration. Clearly, by the definition (1.1), we have that

$$\mathscr{T}(n, k) = \frac{1}{k!} \left. D_t^n \tan^k t \right|_{t=0}.$$

Now, since $T_n = D_t^n \tan t|_{t=0}$, the proposed formula for $\mathscr{T}(n, k)$ in (2.15) could be obtained upon setting $n = k$ in (2.14), then differentiating n times both sides of the resulting expression with respect to t and putting $t = 0$.

In view of Proposition 7, the higher-order tangent numbers $\mathscr{T}(n, k)$ are linear combinations of the tangent numbers T_n as is demonstrated through examples in Example 2.

Example 2. The numbers $\mathscr{T}(n, k)$ expressed through the tangent numbers T_n

$$\mathscr{T}(1,1) = T_1, \quad \mathscr{T}(2,2) = \tfrac{1}{2} T_3 \quad \mathscr{T}(3,1) = T_3, \quad \mathscr{T}(3,3) = -\tfrac{1}{6} T_3 + \tfrac{1}{12} T_5,$$

$$\mathscr{T}(4,2) = \tfrac{1}{2} T_5, \quad \mathscr{T}(4,4) = -\tfrac{1}{18} T_5 + \tfrac{1}{144} T_7,$$

$$\mathscr{T}(5,1) = T_5, \quad \mathscr{T}(5,3) = -\tfrac{1}{6} T_5 + \tfrac{1}{12} T_7, \quad \mathscr{T}(5,5) = \tfrac{1}{120} T_5 - \tfrac{1}{144} T_7 + \tfrac{1}{2880} T_9,$$

$$\mathscr{T}(6,2) = \tfrac{1}{2} T_7, \quad \mathscr{T}(6,4) = -\tfrac{1}{18} T_7 + \tfrac{1}{144} T_9, \quad \mathscr{T}(6,6) = \tfrac{23}{10800} T_7 - \tfrac{1}{2160} T_9 + \tfrac{1}{86400} T_{11},$$

$$\mathscr{T}(7,1) = T_7, \quad \mathscr{T}(7,3) = -\tfrac{1}{6} T_7 + \tfrac{1}{12} T_9, \quad \mathscr{T}(7,5) = \tfrac{1}{120} T_7 - \tfrac{1}{144} T_9 + \tfrac{1}{2880} T_{11},$$

$$\mathscr{T}(7,7) = -\tfrac{1}{5040} T_7 + \tfrac{7}{32400} T_9 - \tfrac{1}{51840} T_{11} + \tfrac{1}{3628800} T_{13},$$

$$\mathscr{T}(8,2) = \tfrac{1}{2} T_9, \quad \mathscr{T}(8,4) = -\tfrac{1}{18} T_9 + \tfrac{1}{144} T_{11}, \quad \mathscr{T}(8,6) = \tfrac{23}{10800} T_9 - \tfrac{1}{2160} T_{11} + \tfrac{1}{86400} T_{13},$$

$$\mathscr{T}(8,8) = -\tfrac{11}{264600} T_9 + \tfrac{11}{907200} T_{11} - \tfrac{1}{1814400} T_{13} + \tfrac{1}{203212800} T_{15}.$$

Proposition 8. *We have:*

$$\mathscr{T}(n,k) = (-1)^{\frac{n-k}{2}}(-1)^n \frac{2^n}{k!} \sum_{\alpha=k}^{n} \sum_{\beta=1}^{\alpha} (-1)^\beta \binom{\alpha-1}{k-1}\binom{\alpha}{\beta}\frac{\beta^n}{2^\alpha} \qquad (n \geq 1, k \geq 0). \qquad (2.16)$$

Demonstration. To prove this formula, it suffices to recall the following power series expansion of $\tan^k x$ which was deduced by Schwatt [9, p. 67, Eq. (76)]

$$\tan^k x = \sum_{n=k}^{\infty}(-1)^{\frac{n-k}{2}}(-1)^n\, 2^n \sum_{\alpha=k}^{n}\sum_{\beta=1}^{\alpha}(-1)^\beta\binom{\alpha-1}{k-1}\binom{\alpha}{\beta}\frac{\beta^n}{2^\alpha}\cdot\frac{x^n}{n!}.$$

Proposition 9. *Let respectively $S(n,k)$ and $L_{n,k}$ be the Stirling numbers of the second kind and Lah numbers defined by [1, p. 50]*

$$\left(e^t - 1\right)^k = k! \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}$$

and [1, p. 156]

$$L_{n,k} = (-1)^n\binom{n-1}{k-1}\frac{n!}{k!}.$$

Then, for $n \geq 1$ and $k \geq 0$, we have that

$$\mathscr{T}(n,k) = (-1)^{\frac{n-k}{2}}(-1)^n \sum_{\alpha=k}^{n}(-1)^\alpha\, 2^{n-\alpha}\, S(n,\alpha)\binom{\alpha-1}{k-1}\frac{\alpha!}{k!} \qquad (2.17)$$

and

$$\mathscr{T}(n,k) = (-1)^{\frac{n-k}{2}}(-1)^n \sum_{\alpha=k}^{n} 2^{n-\alpha}L_{\alpha,k}\, S(n,\alpha). \qquad (2.18)$$

Demonstration. It is well–known that the Stirling numbers of the second kind are given by means of $S(n,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n$ [1, p. 204]. It is clear, then, that the sought formula (2.17) follows straightforwardly by making use of this sum and (2.16). Furthermore, the expression on the right-side of the equation (2.17), in view of the definition of the Lah numbers $L_{n,k}$, may be written in the form given by (2.18).

For the sake of completeness of this paper, the following result is reproduced from the work of Butzer *et al.* [10, Proposition 7.5, p. 482], to which the interested reader is referred for a simple proof.

Proposition 10. *In terms of central factorial numbers $T(n,k)$ which are explicitly given by $T(n,k) = \frac{1}{k!}\sum_{\alpha=0}^{k}(-1)^\alpha\binom{k}{\alpha}\left(\frac{k}{2}-\alpha\right)^n$ (see [10, p. 429] and [11, pp. 213–217]), for $n,k \geq 1$, we have:*

$$\mathscr{T}(2n,2k) = \sum_{\alpha=k}^{n}(-1)^{n-\alpha}\, 2^{2n-2\alpha}\frac{(2\alpha)!}{(2k)!}\binom{\alpha-1}{\alpha-k}T(2n,2\alpha), \qquad (2.19)$$

$$\mathscr{T}(2n+1,2k+1) = \sum_{\alpha=k}^{n}(-1)^{n-\alpha}\, 2^{2n-2\alpha}\frac{(2\alpha+1)!}{(2k+1)!}\binom{\alpha-1/2}{\alpha-k}T(2n+1,2\alpha+1). \qquad (2.20)$$

3. Concluding remarks

In this paper, two sequences of non-negative integer numbers, $\{\mathscr{T}(n,k)\}_{n,k=0}^{\infty}$ and $\{\mathscr{S}(n,k)\}_{n,k=0}^{\infty}$, have been thoroughly investigated and many of their properties are determined. In particular, it should be mentioned that the higher-order tangent numbers $\mathscr{T}(n,k)$ constitute a special class of the partial multivariate Bell polynomials. Furthermore, from the knowledge of $\mathscr{T}(n,k)$, the higher-order secant numbers $\mathscr{S}(n,k)$ can be easily computed since $\mathscr{S}(n,k)$ and $\mathscr{T}(n,k)$ are related (see Proposition 3). In addition, a simple explicit formula involving a double finite sum is deduced for the higher-order tangent numbers $\mathscr{T}(n,k)$ (Proposition 8) and it is shown that $\mathscr{T}(n,k)$ are linear combinations of the classical tangent numbers T_n (Proposition 7).

As an example of an application of the numbers $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$ we give, without proof, recently established [6, Corollaries 1 and 2], and here slightly modified (observe that $T(n,k) = \mathscr{T}(n,k)\,k!$ and $S(n,k) = \mathscr{S}(n,k)\,k!$), closed-form higher derivative formulae.

Proposition 11. *In terms of the tangent and secant numbers of order k, $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$, for $n \geq 0$, we have:*

(a) $\quad D_t^n \tan x = \mathscr{T}(n,1) + \sum_{k=1}^{n+1} (k-1)!\,\mathscr{T}(n+1,k) \tan^k x;$

(b) $\quad D_t^n \sec x = \sec x \sum_{k=0}^{n} k!\,\mathscr{S}(n,k)\,\tan^k x;$

(c) $\quad D_t^n \cot x = (-1)^n \left[\mathscr{T}(n,1) + \sum_{k=1}^{n+1} (k-1)!\,\mathscr{T}(n+1,k)\,\cot^k x \right];$

(d) $\quad D_t^n \csc x = (-1)^n \csc x \sum_{k=0}^{n} k!\,\mathscr{S}(n,k)\,\cot^k x.$

Proposition 12. *In terms of the tangent and secant numbers of order k, $\mathscr{T}(n,k)$ and $\mathscr{S}(n,k)$, for $n \geq 0$, we have:*

(a) $\quad D_t^n \tanh x = (-1)^{\frac{n-1}{2}}\,\mathscr{T}(n,1) + \sum_{k=1}^{n+1} (-1)^{\frac{n+k-1}{2}}\,(k-1)!\,\mathscr{T}(n+1,k) \tanh^k x;$

(b) $\quad D_t^n \operatorname{sech} x = \operatorname{sech} x \sum_{k=0}^{n} (-1)^{\frac{n+k}{2}}\,\mathscr{S}(n,k)\,\tanh^k x;$

(c) $\quad D_t^n \coth x = (-1)^{\frac{n-1}{2}}\,\mathscr{T}(n,1) + \sum_{k=1}^{n+1} (-1)^{\frac{n+k-1}{2}}\,(k-1)!\,\mathscr{T}(n+1,k)\,\coth^k x;$

(d) $\quad D_t^n \operatorname{csch} x = \operatorname{csch} x \sum_{k=0}^{n} (-1)^{\frac{n+k}{2}}\,\mathscr{S}(n,k)\,\coth^k x.$

We remark that, apart from $\mathscr{T}(n,k)$, which were introduced by Comtet (see the definition of $T(n,k)$ in [1, p. 259]), in the literature can be found several more definitions of higher-order tangent numbers: Carlitz and Scoville considered the numbers $T_n^{(k)} := k!\mathscr{T}(n,k)$ ([2, p. 428] and [3, p. 305]), while Lomont studied $C_n^{(k)}$ given by $C_n^{(k)} := (-1)^{\frac{n-k}{2}}(-1)^n k!\,\mathscr{T}(n,k)$ [12, Eq. (3.1)]. However, it seems that numbers $T_n^{(z)}$ defined by Cenkci [13, p. 1500] cannot be brought into connection with $\mathscr{T}(n,k)$. We note also that Carlitz and Scoville [2, p. 428] defined higher-order secant numbers $S_n^{(k)} := k!\mathscr{S}(n,k)$ (see also [4, p. 665]). The tangent and secant numbers, T_n and S_n, were first studied by André in 1879 [14].

The majority of results presented here are new, while the known results are two recurrence relations given by Proposition 5 [for the recurrence (2.8) see [4, Eq. (7)] and [1, p. 258]; for the recurrence (2.9) see [4, p. 665, Eq. (9)]] as well as the formula given by Proposition 6(b) [*cf.* (2.11) and [4, p. 665, Eq. (10)]]. Also, the formula (2.18) was deduced earlier by different arguments [15, p. 156]. In addition, it should be noted that, judging by the results of extensive numerical calculations which support such a conclusion, our simple and rather compact formulae (2.6) and (2.15) appear to be equivalent to the (relatively complicated-looking) formulae which were established by Lomont [12, see, respectively, Eq. (3.2) and Eqs. (3.4), (3.6) and (3.7)].

Acknowledgements

References

[1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.

[2] L. Carlitz and R. Scoville, Tangent numbers and operators, *Duke Math. J.* **39** (1972), 413–429.

[3] L. Carlitz, Permutations, sequences and special functions, *SIAM Review* **17** (1975), 298–322.

[4] D.E. Knuth and T.J. Buckholtz, Computation of tangent, Euler, and Bernoulli numbers, *Math. Comp.*, **21** (1967), 663-688.

[5] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[6] D. Cvijović, Derivative polynomials and closed-form higher derivative formulae, *Appl. Math. Comput.* **215** (2009), 3002–3006.

[7] H.W. Gould, Coefficient identities for powers of Taylor and Dirichlet series, *Amer. Math. Monthly* **81** (1974), 3–14.

[8] C.A. Charalambides, *Enumerative Combinatorics*, Chapman and Hall/CRC, Boca Raton, 2002.

[9] I.J. Schwatt, *An Introduction to the Operations with Series*, University of Pennsylvania Press, Philadelphia, 1924.

[10] P.L. Butzer, M. Schmidt, E.L. Stark and L. Vogt, Central factorial numbers; their main properties and some applications, *Numer. Funct. Anal. Optim.* **10** (1989), 419–488.

[11] J. Riordan, *Combinatorial identities*, John Wiley and Sons, New York, 1968.

[12] J.S. Lomont, Formulas for powers of the hyperbolic tangent with an application to higher-order tangent numbers, *Rocky Mountain J. Math.* **35** (2005), 1217–1231.

[13] M. Cenkci, An explicit formula for generalized potential polynomials and its applications, *Discrete Math.* **309** (2009), 1498–1510.

[14] D. André, Développements de sec x et de tang x, *C.R. Acad. Sc. Paris* **88** (1879), 965–967.

[15] J.S. Lomont and J. Brillhart, *Elliptic Polynomials*, Chapman and Hall and RCC Press, Boca Raton, 2000.